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A Note on the Individual Ergodic Theorem on Product MV Algebras[†]

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Classical ergodic theory was built on σ-algebras. The aim of this paper is to prove the individual ergodic theorem on more general structures—on product MV algebras.

1. INTRODUCTION

The individual ergodic theorem, well known from classical ergodic theory [9], has been proved on a dynamical system $(\Omega, \mathcal{S}, P, T)$, where Ω is a nonempty set, $\mathcal{S}$ is a σ-algebra on Ω, P is a measure on $\mathcal{S}$, and T is a measure-preserving transformation.

This theorem was later solved on more general structures, for example, on fuzzy quantum loqics [10] and on D-posets [4]. This paper deals with the individual ergodic theorem on product MV algebras [3, 8].

2. PRELIMINARIES

MV algebras, introduced by Chang [1], are many-valued analogues of a two-valued logic. An MV algebra is a nonempty set $\mathcal{M}$ with two constant elements $0_{\mathcal{M}}$ and $1_{\mathcal{M}}$ $(0_{\mathcal{M}} \neq 1_{\mathcal{M}})$, with a binary operation $\oplus$ and a unary operation $*$ such that, for all $a, b, c \in \mathcal{M}$, we have:

[MV1] $a \oplus b = b \oplus a$ (commutativity).
[MV2] $(a \oplus b) \oplus c = a \oplus (b \oplus c)$ (associativity).

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[MV3] $a \oplus 0_{\mathcal{M}} = a$.
[MV4] $a \oplus 1_{\mathcal{M}} = 1_{\mathcal{M}}$.
[MV5] $(a^*)^* = a$.
[MV6] $a \oplus a^* = 1_{\mathcal{M}}$.
[MV7] $0^*_{\mathcal{M}} = 1_{\mathcal{M}}$.
[MV8] $(a^* \oplus b)^* \oplus b = (a \oplus b^*)^* \oplus a$.

The lattice operations $\vee$ and $\wedge$ can be defined in any MV algebra by

$$a \bigvee b := (a^* \oplus b)^* \oplus b$$
$$a \bigwedge b := (a^* \bigvee b^*)^*, \qquad a, b, \in \mathcal{M}$$

If, for $a, b \in \mathcal{M}$, we define

$$a \leq b \Leftrightarrow a = a \bigwedge b$$

then $\leq$ is a partial order on $\mathcal{M}$ and the MV algebra is a distributive lattice with respect to the operations $\vee$ and $\wedge$. We recall that $a \leq b$ iff $b \oplus a^* = 1_{\mathcal{M}}$. We can define the binary operations $\odot$ and $-$ as follows [2]:

$$a \odot b := (a^* \oplus b^*)^*$$
$$a - b := (a^* \oplus b)^*, \qquad a, b \in \mathcal{M}$$

An MV algebra $\mathcal{M}$ is called an *MV* σ-*algebra* if each countable sequence of elements from $\mathcal{M}$ has the supremum in $\mathcal{M}$. One of the important Mundici results [7] says that any MV algebra can be represented by a commutative l-group $(\mathcal{G}, +, 0, \leq)$ with a strong unit u, i.e., for any $a \in \mathcal{G}$, there exists an integer $n \geq 1$ such that $a \leq nu$. In any MV algebra $\mathcal{M}$, we can introduce a partial binary operation $+$ defined iff $a \leq b^*$ via

$$a + b := a \oplus b$$

A *product MV algebra* [8] is an algebraic system $(\mathcal{M}, \oplus, \cdot, *, 1, 0)$, where $(\mathcal{M}, \oplus, *, 1, 0)$ is an MV algebra and $\cdot$ is a binary operation satisfying following conditions:

[P1] $1_{\mathcal{M}} \cdot 1_{\mathcal{M}} = 1_{\mathcal{M}}$.
[P2] The operation $\cdot$ is commutative and associative.
[P3] If $a + b \leq 1_{\mathcal{M}}$, then $c \cdot (a + b) = c \cdot a + c \cdot b$, $a, b, c \in \mathcal{M}$.
[P4] If $a_n \searrow 0$, $b_n \searrow 0$, then $a_n \cdot b_n \searrow 0$.

An observable is a mapping $x: \mathcal{B}(\mathbb{R}) \to \mathcal{M}$ such that $x(\mathbb{R}) = 1_{\mathcal{M}}$, $x(A \cup B) = x(A) + x(B)$ whenever $A \cap B = \emptyset$ and $x(A_n) \nearrow x(A)$ whenever $A_n \nearrow A$.

A state is a mapping $m: \mathcal{M} \to \langle 0, 1 \rangle$ such that $m(1_{\mathcal{M}}) = 1$, $m(a \oplus b) = m(a) + m(b)$ whenever $a \leqq b^*$ and $m(a) = \lim_{n\to\infty} m(a_n)$ whenever $a_n \nearrow a$.

The mapping m_x: $\mathcal{B}(\mathbb{R}) \to \langle 0, 1\rangle$ defined by the formula $m_x(A) = m(x(A))$ is a probability measure [6]. An observable x is an *integrable observable* [10] if the integral $\int_{\mathrm{R}} t\, dm_x(t)$ exists. In this case the mean value of the observable x is defined via $E(x) = \int_{\mathrm{R}} t\, dm_x(t)$.

Let x, y: $\mathcal{B}(\mathbb{R}) \to \mathcal{M}$ be two observables. The joint observable [8] of the observables x, y is a mapping h: $\mathcal{B}(\mathbb{R}^2) \to \mathcal{M}$ satisfying following conditions:

[JO1] $h(\mathbb{R}^2) = 1$.
[JO2] If $A \cap B = \emptyset$, then $h(A \cup B) = h(A) + h(B)$.
[JO3] If $A_n \nearrow A$, then $h(A_n) \nearrow h(A)$.
[JO4] $h(C \times D) = x(C) \cdot y(D)$, $C, D \in \mathcal{B}(\mathbb{R})$.

If $\mathcal{M}$ is a σ-complete, weakly σ-distributive product MV algebra [i.e., every bounded countable subset of $\mathcal{M}$ has the supremum and for any bounded double sequence $(a_{i,j})_{i,j} \subset \mathcal{M}$ such that for $a_{i,j} \searrow 0$ ($j \to \infty$, $i = 1, 2, \ldots$) it holds that $\wedge_{\phi \in \mathbb{N}^{\mathbb{N}}} \vee_{i=1}^{\infty} a_{i\phi(i)} = 0$], then for any observables x, y there exists their joint observable [8]. We can also generalize this result.

Let $J \subset \mathbb{N}$, $J = \{i_1, \ldots, i_k\}$ and let $x_{i_1}, \ldots, x_{i_k}$ be observables on $\mathcal{M}$. Then there exists a mapping h_J: $\mathcal{B}(\mathbb{R}^{|J|}) \to \mathcal{M}$ such that the following properties are satisfied:

[J1] $h_J(\mathbb{R}^{|J|}) = 1_{\mathcal{M}}$.
[J2] If $A \cap B = \emptyset$, then $h_J(A \cup B) = h_J(A) + h_J(B)$.
[J3] If $A_n \nearrow A$, then $h_J(A_n) \nearrow h_J(A)$.
[J4] $h_J(A_{i_1} \times \cdots \times A_{i_k}) = x_{i_1}(A_{i_1}) \cdot \cdots \cdot x_{i_k}(A_{i_k})$, $A_{i_1}, \ldots, A_{i_k} \in \mathcal{B}(\mathbb{R})$.

To solve some problems of probability theory on this structure, it seems to be necessary to assume that the operation $\cdot$ has the next property:

[P5] $a \cdot 1_{\mathcal{M}} = a$ for all $a \in \mathcal{M}$.

Moreover, for the sake of the individual ergodic theorem we need to work not only with observables, but also with the composite mapping of observables. By using the joint observable we are able to construct some operations with observables. For example,

$$\frac{1}{n}\sum_{i=1}^{k} x_i := h_j \circ g^{-1}, \qquad \text{where} \quad g\colon \mathbb{R}^{|J|} \to \mathbb{R}; \qquad g(u_1, \ldots, u_k) = \frac{1}{k}\sum_{i=1}^{k} u_i$$

In the following result, by $\mathcal{M}$ we will denote the σ-complete, weakly σ-distributive product MV algebra with properties [P1]–[P5].

Lemma 1. A mapping h_J: $\mathcal{B}(\mathbb{R}^{|J|}) \to \mathcal{M}$ satisfying [J1]–[J4] has the following properties:

[J5] If $A \in \mathcal{B}(\mathbb{R})$, then $h_J((t_1, \ldots, t_i, \ldots, t_k) \in \mathbb{R}^{|J|}, t_i \in A) = x_i(A)$.

[J6] If $J_1 \subset J_1 \subset \mathbb{N}$, then $h_{J_2}(\pi^{-1}_{J_2J_1}(A)) = h_{J_1}(A)$ for all $A \in \mathcal{B}(\mathbb{R}^{|J_1|}$, where $\pi_{J_2J_1}$: $\mathcal{B}(\mathbb{R}^{|J_2|})\mathcal{B}(\mathbb{R}^{|J_1|})$ is the projection.

Proof. [J5] we have

$$\begin{aligned} &h_J(\{(t_1, \ldots, t_i, \ldots, t_k) \in \mathbb{R}^{|J|}, t_i \in A\}) \\ &\quad = h_J(\mathbb{R} \times \cdots \times \mathbb{R} \times A \times \mathbb{R} \cdots \times \mathbb{R}) \\ &\quad = x_1(\mathbb{R}) \cdot \cdots \cdot x_{i-1}(\mathbb{R}) \cdot x_i(A) \cdot x_{i+1}(\mathbb{R}) \cdot \cdots \cdot x_k(\mathbb{R}) \\ &\quad = 1_{\mathcal{M}} \cdot \cdots \cdot 1_{\mathcal{M}} \cdot x_i(A) \cdot 1_{\mathcal{M}} \cdot \cdots \cdot 1_{\mathcal{M}} \\ &\quad = x_i(A) \end{aligned}$$

[J6] Let $J_1 \subset J_2 \subset \mathbb{N}$, $A = A_{t_1} \times \cdots \times A_{t_k} \in \mathcal{B}(\mathbb{R}^{|J_1|})$ and

$$\pi^{-1}_{J_2J_1}(A) = \mathbb{R} \times \cdots \times \mathbb{R} \times A_{t_1} \times \cdots \times A_{t_k} \times \mathbb{R} \times \cdots \times \mathbb{R} \in \mathcal{B}(\mathbb{R}^{|J_2|})$$

Then

$$\begin{aligned} &h_{J_2}(\pi^{-1}_{J_2J_1}(A)) \\ &\quad = x_{s_1}(\mathbb{R}) \cdot \cdots \cdot x_{s_i}(\mathbb{R}) \cdot x_{t_1}(A_{t_1}) \cdot \cdots \cdot x_{t_k}(A_{t_k}) \cdot x_{s_j}(\mathbb{R}) \cdot \cdots \cdot x_{s_n}(\mathbb{R}) \\ &\quad = 1_{\mathcal{M}} \cdot \cdots \cdot 1_{\mathcal{M}} \cdot x_{t_1}(A_{t_1}) \cdot \cdots \cdot x_{t_k}(A_{t_k}) \cdot 1_{\mathcal{M}} \cdot \cdots \cdot 1_{\mathcal{M}} \\ &\quad = h_{J_1}(A) \end{aligned}$$

Let us put $\mathcal{L} = \{A \in \mathcal{B}\ (\mathbb{R}^{|J_1|});\ h_{J_2}(\pi^{-1}_{J_2J_1}(A)) = h_{J_1}(A)\}$ and denote by $\mathcal{D}$ the family of all rectangles $A_{t_1} \times \cdots \times A_{t_k}$; $A_{t_1}, \ldots, A_{t_k} \in \mathcal{B}(\mathbb{R})$. Evidently, $\mathcal{L} \supset \mathcal{D}$. From the properties of the mapping h_J it follows that $\mathcal{L}$ is a $q - \sigma$-algebra over the ring $\mathfrak{S}(\mathcal{D})$ generated by $\mathcal{D}$. Therefore

$$\mathcal{L} \supset q - \sigma(\mathfrak{S}(\mathcal{D})) = \sigma(\mathfrak{S}(\mathcal{D})) = \mathcal{B}(\mathbb{R}^{|J_1|})$$

which implies

$$h_{J_2}(\pi^{-1}_{J_2J_1}(A)) = h_{J_1}(A)$$

whenever $A \in \mathcal{B}\ (\mathbb{R}^{|J_1|})$. ■

Let m be a state on $\mathcal{M}$. According to the property [J6] of the mapping h_J, we are able to define the consistency system of the probability measures $\{P_J, \emptyset \neq J \subset \mathbb{N}, J \text{ is finite}\}$ defined via

$$P_J(A) = m(h_J(A)),\ A \in \mathcal{B}(\mathbb{R}^{|J|})$$

It is not difficult to prove that P_J is a probability measure and it holds that

$$P_{J_1}(A) = P_{J_2}(\pi^{-1}_{J_2J_1}(A))$$

where $\pi_{J_2J_1}$: $\mathbb{R}^{|J_2|} \to \mathbb{R}^{|J_1|}$ is the projection, $J_1 \subset J_2 \subset \mathbb{N}$, $A \in \mathcal{B}(\mathbb{R}^{|J_1|})$.

Let $\mathbb{R}^{\mathbb{N}}$ be the set of all sequences of real numbers, and π_J: $\mathbb{R}^{\mathbb{N}} \to \mathbb{R}^{|J|}$ be the projection, i.e., $\pi_J((t_n)_n) = (t_{j_i}, \ldots, t_{j_k})$ for $J = \{j_1, \ldots, j_k\}$. By the Kolmogorov theorem there exists exactly one probability measure P on the measurable space $(\mathbb{R}^{\mathbb{N}}, \sigma(C))$ [C is the family of all cylinders, i.e., the set of all sets of the form $\pi_J^{-1}(A)$, $J \subset \mathbb{N}$, $A \in \mathcal{B}(\mathbb{R}^{|J|})$ and $\sigma(C)$ is the σ-algebra over C] satisfying the equality

$$P(\pi_J^{-1}(A)) = P_J(A)$$

for all $A \in \mathcal{B}(\mathbb{R}^{|J|})$ and every finite $J \subset \mathbb{N}$.

Let us define the mapping ξ_i: $\mathbb{R}^{\mathbb{N}} \to \mathbb{R}$; $\xi_i((t_n)_n) = t_i$, $i = 1, 2, \ldots$. Evidently, ξ_i is a random variable and it holds that

$$\begin{aligned} P_{\xi_i}(A) = P(\xi_i^{-1}(A) &= P(\{(t_n)_n; t_i \in A\}) \\ &= P(\pi_{\{i\}}^{-1}(A)) \\ &= m(h_{\{i\}}(A)) \\ &= m(x_i(A)) \\ &= m_{x_i}(A) \end{aligned}$$

We showed that for any sequence $(x_n)_n$ of observables we can construct a sequence $(\xi_n)_n$ of random variables.

In ref. 10, 8.6, there is a modification of almost everywhere convergence with the help of lim-sup and lim-inf. This modification gives the possibility to solve the individual ergodic theorem on $\mathcal{M}$. An important result which refers to problems of upper and lower limits of sequences of observables on $\mathcal{M}$ is the next theorem.

Theorem 2 [10, Theorem 8.6.9]. Let $(x_n)_n$ be a sequence of observables, $(\xi_n)_n$ be a sequence of corresponding projections, and $(g_n)_n$ be a sequence of Borel measurable functions g_n: $\mathbb{R}^n \to \mathbb{R}$. If $(g_n(\xi_1, \ldots, \xi_n))_n$, converges P-almost everywhere, then $(g_n(x_1, \ldots, x_n))_n$ converges m-almost everywhere, i.e., there exists $\bar{x} = \limsup_{n\to\infty} g_n(x_1, \ldots, x_n)$ and $\underline{x} = \liminf_{n\to\infty} g_n(x_1, \ldots, x_n)$ and

$$\begin{aligned} &m(\limsup_{n\to\infty} g_n(x_1, \ldots, x_n)(-\infty, t)) \\ &\quad = m(\liminf_{n\to\infty} g_n(x_1, \ldots, x_n)(-\infty, t)) \end{aligned}$$

for every t. Moreover,

$$P(\{u \in \mathbb{R}^{\mathbb{N}}: \limsup_{n\to\infty} g_n(\xi_1(u), \ldots, \xi_n(u)) < t\})$$
$$= m(\limsup_{n\to\infty} g_n(x_1, \ldots, x_n)(-\infty, t))$$

for every $t \in \mathbb{R}$.

Definition 3. A mapping $\tau: \mathcal{M} \to \mathcal{M}$ is an m-preserving transformation if the following conditions are satisfied:

[T1] $\tau(1_{\mathcal{M}}) = 1_{\mathcal{M}}$.
[T2] If $a + b \leq 1$, then $\tau(a + b) = \tau(a) + \tau(b)$, $a, b \in \mathcal{M}$.
[T3] If $a_n \nearrow a$, then $\tau(a_n) \nearrow \tau(a)$, $a_n \in \mathcal{M}$ for all $n \in \mathbb{N}$.
[T4] $m(\tau(a) \cdot \tau(b)) = m(a \cdot b)$, $a, b \in \mathcal{M}$.

3. INDIVIDUAL ERGODIC THEOREM

Theorem 4. Let x be an integrable observable on $\mathcal{M}$. Let $\tau: \mathcal{M} \to \mathcal{M}$ be an m-preserving transformation and let $\tau^i \circ x$ be bounded, i.e., there exist observables y, z such that

$$y((-\infty, t)) \leq \tau^i \circ x((-\infty, t)) \leq z((-\infty, t))$$

for every $t \in \mathbb{R}$ and every $i \in \mathbb{N}$. Then there exists an observable x^* with the following properties:

[E1] $E(x) = E(x^*)$.
[E2] $(1/n) \sum_{i=0}^{n-1} \tau^i \circ x \to x^*$ almost everywhere in the state m.

Proof. Let $x_n = \tau^{n-1} \circ x$, $n = 1, \ldots$. We return to the probability space $(\mathbb{R}^{\mathbb{N}}, \sigma(C), P)$ [$\sigma(C)$ is the σ-algebra over the family of all cylinders in $\mathbb{R}^{\mathbb{N}}$ such that

$$P(\{(t_i)_i; t_i \in A_i, i = 1, \ldots, n\}) = m(x_1(A_1) \cdot \cdots \cdot x_n(A_n))$$

for any $A_i \in \mathcal{B}(\mathbb{R})$]. Let $T: \mathbb{R}^{\mathbb{N}} \to \mathbb{R}^{\mathbb{N}}$ be the shift defined by the formula

$$T((t_n)_n) = (s_n)_n, \qquad \text{where} \quad s_n = t_{n+1} \quad \text{for all} \quad n \in \mathbb{N}$$

Let $A = \{(t_i)_i: t_{i_1} \in A_1, \ldots, t_{i_k} \in A_k\}$ be the cylinder. In this case

$$T^{-1}(A) = \{((t_i)_i; T((t_i)_i) \in A\}$$
$$= \{(t_i)_i; t_{i_1+1} \in A_1, \ldots, t_{i_k+1} \in A_k\}$$

Therefore

$$P(T^{-1}(A)) = m(x_{i_1+1}(A_1) \cdot \cdots \cdot x_{i_k+1}(A_k))$$
$$= m(\tau^{i_1}(x(A_1)) \cdot \cdots \cdot \tau^{i_k}(x(A_k)))$$

$$= m(\tau^{i_1-1}(x(A_1)) \cdot \cdots \cdot \tau^{i_k-1}(x(A_k)))$$
$$= m(x_{i_1}(A_1) \cdot \cdots \cdot x_{i_k}(A_k))$$
$$= P(A)$$

We showed that T preserves the probability measure, i.e.,

$$P((T^{-1}(A)) = P(A)$$

Since x_1 is intergrable, the first coordinate of function ξ_1 [defined by $\xi_i((t_i)_i) = t_i$] is integrable, too. Therefore by the individual ergodic theorem [9] there exists an integrable random variable ξ^* such that $E(\xi^*) = E(\xi_1)$ and

$$\frac{1}{n}\sum_{i=0}^{n-1} \xi_1 \circ T^i \to \xi^* \qquad P\text{-almost everywhere}$$

Of course, $\xi_1 \circ T_i = \xi_{i+1}$ and

$$\frac{1}{n}\sum_{j=1}^{n} \xi_j \to \xi^* \qquad P\text{-almost everywhere.}$$

Put $g_n(u_1, \ldots, u_n) = (1/n) \sum_{i=1}^n u_i$. According to Theorem 2, the sequence

$$(g_n(x_1, \ldots, x_n))_n = \left(\frac{1}{n}\sum_{i=1}^{n} x_i\right)_n = \left(\frac{1}{n}\sum_{i=0}^{n-1} \tau^i \circ x\right)_n$$

converges m-almost everywhere to $x^* = \limsup_{n\to\infty} g_n(x_1, \ldots, x_n)$ and

$$P(\xi^{*-1}((-\infty, t))) = m\,(x^*((-\infty, t)))$$

for every $t \in \mathbb{R}$. Since $P_{\xi^*} = m_{x^*}$ and $P_{\xi_1} = m_{x_1} = m_x$, we have

$$E(x) = E(\xi_1) = E(\xi^*) = E(x^*) \quad \blacksquare$$

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